Regularization, Optimization and Approximation in General Hausdorff Topological Spaces

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Complexity of real World and Modelling

Features of complex systems (biological systems, oceanology, geology, physics, economics and sociology...etc):

- openness,
- fluctuation,
- chaos,
- disorder,
- blur,
- creativity,
- contradiction,
- ambiguity,
- paradox,
- instability

Complexity of real World and Modelling

Albert Einstein:

"if we do not change our way of analyzing we will not be able to solve the problems we create with our current ways of thinking"

But this new way of thinking has a name: Systemic approach or mathematical modelling

"si nous ne changeons pas notre façon d'analyser les phénomènes, nous ne serons pas capables de résoudre les problèmes que nous créons avec nos modes actuels de pensée"

Or cette nouvelle manière de penser a un nom: Approche systémique ou la modélisation mathématique.

Stable models (Classical Mechanics,Scientific positivism,17-19 century)

- A model is said to be stable if small perturbations at its parameters lead to small perturbations in its solutions
- if the measurement errors at its parameters are proportional to the measurement errors in the solutions

Example of stable model:

$$\sum_i \overrightarrow{F_i(t)} = m.\frac{d^2\overrightarrow{X}}{dt^2}(t),$$

$$\frac{d\overrightarrow{X}}{dt}(t_0) = \overrightarrow{V_0},$$

$$\overrightarrow{X(t_0)} = \overrightarrow{X_0}$$

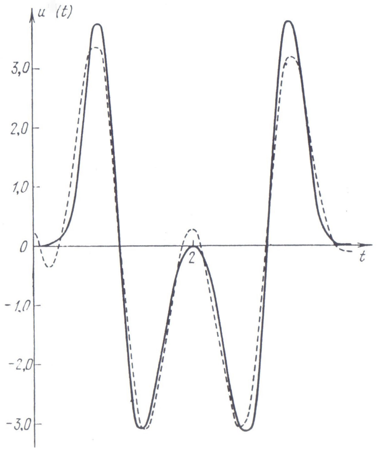
Unstable models (J.Hadamard 1903, H.Poincaré, Edward Lorenz 1961, A.Tikhonov 1963)

- A model is said to be unstable if small measurement errors in its parameters lead to uncontrollable measurement errors in its solutions.
- Nuclear physics, signal theory, inverse problems, image analysis, geophysics, optimal control and PDE theory.

Unstable models: Examples

Signal theory, Spectroscopy, Nuclear physics:

- $z(s) \to \boxed{\Psi(x,s)} \to u(x)$
- Curve :



Unstable models: Examples

The problem of studying the spectral composition of a beam of light:
Suppose that the observed radiation is non-homogeneous and that the distribution of the energy density over the spectrum is characterized by a function $z(s)$ which s is the frequency. If we pass the beam through a measuring apparatus you obtain an experimental spectrum u(x), here x may be the frequency and it may also be expressed in terms of voltage or current of the measuring device.

$$Az = \int_a^b z(s)\Psi(x,s)ds = u(x), x \in [c,d]$$ **(Theorical model):**

$$Az_\epsilon = u_\epsilon$$ **(Approximated model)**

$$u_\epsilon(x) = \int_a^b z_\epsilon(s)\Psi(x,s)ds$$

$$\int_a^b z_1(s)\Psi(x,s)ds = u_1(x), x \in [c,d]$$

And

$$z_\epsilon(s) = z_1(s) + N.sin(\tfrac{s}{\epsilon})$$

Unstable models: Examples

It is clear that z_ϵ is a solution of:

$$u_\epsilon(x) = \int_a^b z_\epsilon(s)\Psi(x,s)ds = u_1(x) + N.\int_a^b sin(\frac{s}{\epsilon})\Psi(x,s)ds$$

$$\| u_\epsilon - u_1 \|_{L^2[c,d]} \to 0, \epsilon \to 0 \forall N$$

But:

$$\| z_\epsilon - z_1 \|' \nrightarrow 0, \epsilon \to 0$$

In the two cases where:

$$\| z_\epsilon - z_1 \|' = max_{s\in[a,b]} \mid z_\epsilon(s) - z_1(s) \mid = N$$

Or

$$\| z_\epsilon - z_1 \|' = \| z_\epsilon - z_1 \|_{L^2[a,b]} = N(\frac{b-a}{2} - \frac{\epsilon}{2}sin(\frac{b-a}{2})cos(\frac{b+a}{2}))^{\frac{1}{2}}$$

$$\text{If } u_\epsilon \notin Im(A) : \int_a^b z_\epsilon(s)\Psi(x,s)ds = u_\epsilon(x),\ x \in [c,d], S = \emptyset$$

Unstable models: Examples

We use the notion of quasi-solution: Find $z_\epsilon \in H$ (H is taken from practical considerations) such that:

$$min_{z \in H} \parallel Az - u_\epsilon \parallel = \parallel Az_\epsilon - u_\epsilon \parallel$$

where $\parallel . \parallel$ is a specified norm and H is a specified space.
In this case if:

$$\parallel u_\epsilon - u_1 \parallel \to 0, \epsilon \to 0$$

It is not true that:

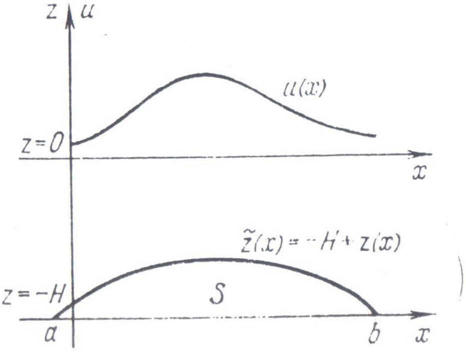
$$\parallel z_\epsilon - z_1 \parallel' \to 0, \epsilon \to 0.$$

The problem of finding $z \in H$ such that $min_{v \in H} \parallel Av - u \parallel = \parallel Az - u \parallel$ is more general to find $z \in H$ such that

$$\int_a^b z(s)\Psi(x,s)ds = u(x), x \in [c,d]$$

Unstable models: Examples

Geophysics:



Unstable models: Examples

Theoretical model:

$$\int_a^b Log(\frac{(x-\zeta)^2+H^2}{(x-\zeta)^2+Z^2(\zeta)})d\zeta = \frac{2\pi}{\rho}\Delta g(x) = \frac{2\pi}{\rho}(g+u(x)-g) = \frac{2\pi}{\rho}u(x)$$

If $\alpha_n(x) \to u(x)$ in a certain sense, in general $Z_n(x) \nrightarrow Z(x)$ with:
$\int_a^b Log(\frac{(x-\zeta)^2+H^2}{(x-\zeta)^2+Z_n^2(\zeta)})d\zeta = \alpha_n(x)$

Unstable models: Examples

Partial differential equations:

- First model in chaos theory: Edward Lorenz 1961:

$$\frac{dx}{dt}(t) = Pr(y - x) \ ,$$

$$\frac{dy}{dt}(t) = -xz + Rx - y \ ,$$

$$\frac{dz}{dt}(t) = xy - bz \ ,$$

$$x(t_0) = x_0 \ ,$$

$$y(t_0) = y_0 \ ,$$

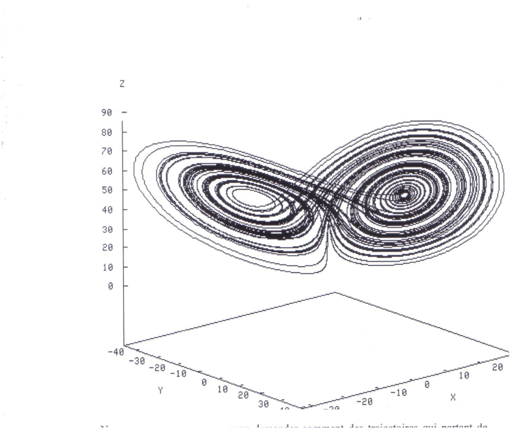
$$z(t_0) = z_0$$

The variable x, y are respectively proportional to the amplitudes of the velocity field and the temperature field while z is connected to the vertical mode temperature, t is time. If:

$$x_\epsilon(t_0) = x_0 + \epsilon \ , \ y_\epsilon(t_0) = y_0 + \epsilon \ , \ z_\epsilon(t_0) = z_0 + \epsilon \ , \ \epsilon \approx 0$$

then $sup_{t \geq T} \parallel (x(t), y(t), z(t)) - (x_\epsilon(t), y_\epsilon(t), z_\epsilon(t)) \parallel \geq \delta(T)$

Unstable models: Examples



- $\frac{\partial u}{\partial y}(x,0) = \varphi(x), u(x,0) = f(x), x \in \Re$ **(Linear PDE)**

$$f_1(x) = \varphi_1(x) = 0, \forall x, \text{ then } u_1(x,y) = 0.$$

If $f_2(x) = 0, \varphi_2(x) = \frac{1}{a} sin(ax)$, then $u_2(x,y) = \frac{1}{a^2} sin(ax).(\frac{e^{ay} - e^{-ay}}{2})$

Unstable models: Examples

It is clear that $\| f_1 - f_2 \| = 0$,
$$\| \varphi_1 - \varphi_2 \| = sup_x \mid \varphi_1(x) - \varphi_2(x) \mid = \frac{1}{a} \to 0 \text{ when } a \to +\infty$$

but

$$\| u_1 - u_2 \| = sup_{x,y} \mid u_1(x,y) - u_2(x,y) \mid = +\infty \ , \forall a$$

$$(f_1, \varphi_1) \sim (f_2, \varphi_2) \text{ if } a \to +\infty \text{ but } \| u_1 - u_2 \| = +\infty, \forall a$$

Instability in Optimization

- $(P) : Min_{x \in C} f(x), f : X \to \Re \cup \{+\infty\}, argmin(f, C) = \{z \in C / f(z) = Min_{x \in C} f(x)\}$
- $(P_\epsilon) : \epsilon.argmin(f, C) = \{z \in C / f(z) \leq Min_{x \in C} f(x) + \epsilon\}$ is the perturbed problem of (P) and $(P_{\epsilon=0}) = (P)$

We say that (P) is unstable or ill-posed if ϵ.**argmin(f,C)** $\nrightarrow$ **argmin(f,C)** if $\epsilon \to 0$ that is: $\exists (x_\epsilon)_\epsilon$ in C such that $x_\epsilon \in \epsilon.argmin(f, C)$ that is $f(x_\epsilon) \to Min_{x \in C} f(x)$ **but** $(x_\epsilon)_\epsilon$ **does not converge to any point in** $argmin(f, C)$

Example: $f(x) = x$ if $x > 0$ and $f(x) = \mid x + 1 \mid$ if $x \leq 0$. We have $f(\frac{1}{n}) \to f(-1) = 0 = min(f)$ but $\frac{1}{n} \nrightarrow -1$.

Instability in Optimization

Definition(A.Tikhonov, 1977):
we say that (P) : $Min_{x \in C} f(x)$ is well-posed in the Tikhonov sense if (P) has a unique solution x' and for any sequence $(x_\epsilon)_\epsilon$ in C such that $f(x_\epsilon) \to Min_{x \in C} f(x)$ if $\epsilon \to 0$ then $(x_\epsilon)_\epsilon \to x'$. So every numerical method generating a minimizing sequence converges to a solution of (P).

If $C = X$ a normed space then (P) is well posed
$\Leftrightarrow f^*(y) = sup_x\{\langle x, y\rangle - f(x)\}$ is differentiable at 0 and $\nabla f^*(0) = 0$.

Instability in Optimization: Examples

- 1) if $f : \Re^n \to \Re$ is convex function with a unique minimizer on $\Re^n$ then $min_{x \in \Re^n} f(x)$ is well-posed.
- 2) ill and well-posedness in optimal control:
  - $min_{u \in C = B_{L^\infty}(0,1)} \{I(u) = \int_0^1 x^2(u)dt\}$ s.b.t $\dot{x} = u$ in $(0,1)$ and $x(0) = 0$, $u \in B_{L^\infty}(0,1) \subset L^\infty(0,1)$

    is ill-posed in the Tikhonov sense because $I(u_\epsilon) \to I(0) = min_C I(u)$, $u_\epsilon(t) = sin(\frac{t}{\epsilon})$ but $u_\epsilon \nrightarrow 0$ in $L^\infty(0,1)$ because $\| u_\epsilon \|_{L^\infty(0,1)} = 1$
  - $min_{u \in C = B_{L^\infty}(0,1)} \{J(u) = \int_0^1 x^2(u)dt + \epsilon . \int_0^1 u^2 dt\}$ s.b.t $\dot{x} = u$ in $(0,1)$ and $x(0) = 0$, $u \in B_{L^\infty}(0,1) \subset L^\infty(0,1)$

    is well-posed $\forall \epsilon > 0$

Regularizing an unstable Problem (P)

Why do we regularize an unstable Problem (P)?

- If (P) is unstable it gives meaningless interpretations in practice
- If there is a lake of good properties as stability, differentiability, convexity, ... etc.

We regularize or stabilize an unstable problem (P) by replacing it by a close robust problem (P_ϵ):

- (P_ϵ) has a unique solution
- (P_ϵ) possesses regular, rich properties at the theoretical or numerical level
- (P_ϵ) provides good interpretations and avoid us a meaningless analysis
- To (P_ϵ) we apply a large class of numerical methods which may be excluded by (P)
- if $\epsilon \to 0$ a solution of (P_ϵ) is a good approximation of a solution of (P)

Regularizing an unstable Problem (P): Examples

In convex optimization:

(P): $Min_{x \in C} f(x)$ is supposed ill-posed in the Tikhonov sense (f, C are convex in R^n, f is continuous, C is closed); that is $\exists (x_\epsilon)_\epsilon$ in C such that $x_\epsilon \in \epsilon argmin(f, C)$, $f(x_\epsilon) \to Min_{x \in C} f(x)$ but $(x_\epsilon)_\epsilon$ does not converge to any point in $S = argmin(f, C) \neq \emptyset$.

(Pϵ) : $Min_{x \in C}(F_\epsilon(x) = f(x) + \epsilon \parallel x - x_0 \parallel^2)$, x_0 is any given point in C.

- (P$_\epsilon$) is well-posed in the Tikhonov sense (stable)
- (P$_\epsilon$) has a unique solution x_ϵ and $x_\epsilon \to proj_S x_0 \in S = argmin(f, C)$
- any algorithm generating a minimizing sequence $f(x_{n,\epsilon}) \to min(P_\epsilon)$ satisfies $(x_{n,\epsilon})_n \to x_\epsilon$ when $n \to +\infty$
- $x_{n,\epsilon}$ is a good approximation of a solution of (P) if n is large enough and ϵ is sufficiently small

Regularization Methods in optimization

- $\mathsf{F}_\epsilon(x) = f(x) + \epsilon \parallel x - x_0 \parallel^p$, $p \geq 2$, $\forall x_0 \in C$
- $\mathsf{F}_\epsilon(x) = f(x) + \epsilon\varphi(x, \epsilon)$ for a suitable choice of φ
- $\mathsf{F}_\epsilon(x) = f(x) + \epsilon \sum_i e^{\frac{1}{\epsilon} g_i(x)}$ (nice properties of the interior barrier method and of the exterior penalty method)
- $\mathsf{F}_\lambda(x) = inf_u\{f(u) + \frac{1}{2\lambda} \parallel x - u \parallel^p\}$, $p \geq 2$ (Moreau-Yoshida regularization of parameter λ and order p) . **F_λ is always C^1 (Frechet differentiability)** if f is convex on a reflexive space. **(f is not necessarily smooth)**
- $argmin(F_\lambda, X) = argmin(f, X)$

Regularization Methods in optimization

- $G_\lambda(x) = inf_u\{f(u) + \Phi(\frac{x-u}{\lambda})\}$ where $\Phi : X \to \Re$ is continuous coercive convex kernel, bounded on bounded sets and X is a normed space
- $L_\lambda(x, y) = min_{u\in X} max_{v\in Y}\{L(u, v) + \frac{1}{2\lambda} \| x-u \|^2 - \frac{1}{2\mu} \| y-v \|^2\}$.$X$ and Y are Hilbert spaces.
  This regularization is used to find a saddle point of the convex-concave function $L : XxY \to \Re$ that is a point (x', y') such that

  $$\forall(x, y) \in XxY, L(x', y) \leq L(x', y') \leq L(x, y') \text{ then}$$
  $$min_x max_y L(x, y) = max_y min_x L(x, y) = L(x', y')$$

  (Mathematical economics, equilibrium problem, location problems, game theory, ...etc)

  - Consider the following saddle problem (Q):$min_{x\in X} max_{y\in Y} L(x, y)$
  - $L_\lambda(x, y)$ is a robust regularization in the sense that we can construct efficient algorithms converging to a solution of (Q) as follows: Given any point $(x_0, y_0) \in XxY$ and set $J_\lambda(x, y) = (x_\lambda, y_\lambda) = argmin_{u\in X} max_{v\in Y}\{L(u, v) + \frac{1}{2\lambda} \| x - u \|^2 - \frac{1}{2\lambda} \| y - v \|^2\}$
  - Consider the following algorithm $(x_{k+1}, y_{k+1}) = J_{\lambda_k}(x_k, y_k)$, $\lambda_k \to 0$. Under a wide class of hypotheses the sequence $(x_k, y_k)_k$ converges to a solution of (Q) from any initial point $(x_0, y_0) \in XxY$.

Regularization in functional analysis and operator theory

- A sequence of functions $\rho_n : \Re^N \to \Re^+$ is said to be a regularizante sequence if $\rho_n \in C_c^\infty(\Re^N)$, $supp(\rho_n) \subset B(0, \frac{1}{n})$, $\int_{\Re^N} \rho_n dx = 1$.
- Example: $\rho_n(x) = Cn^N \rho(nx)$ with $\rho(x) = (e^{\|x\|^2-1})^{-1}$ if $\| x \| < 1$, $\rho(x) = 0$ if $\| x \| \geq 1$, $C = (\int_{\Re^N} \rho dx)^{-1}$
- We can make good regularization by using $(\rho_n)_n$ and the convolution product: $(\rho_n * f)(x) = \int_{\Re^N} \rho_n(t) f(x-t) dt$
- if $f \in L^1_{loc}(\Re^N)$ then $\rho_n * f \in C^\infty(\Re^N)$
- $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, Ω is an open set of $\Re^N$, $p \in [1, +\infty[$
- Frechet-Kolmogorov theorem
- Friedrichs theorem: $\overline{C_c^\infty(\Re^N)/\Omega} = W^{1,p}(\Omega)$, $p \in [1, +\infty[$

Robustness of convergences and approximations

- Let $f : \Re^N \to \overline{\Re}$ and (P): $min_{x \in C} f$. It is well-known that point-wise convergence **is a bad tool in optimization that is:**

$$\text{if } f_n(x) \to f(x) \text{ when } n \to +\infty$$
$$\forall x \in C = \bigcap_n \bigcup_{k \geq n} C_k = \bigcup_n \bigcap_{k \geq n} C_k, \text{ in general}$$

$$min_{C_n} f_n \nrightarrow min_{x \in C} f$$

- uniform convergence is very strong and not in general satisfied in practice.
  Taking in account that the regularization functions are robust we can show in the convex case that

$$F_\lambda^n(x) = inf_u\{f_n(x) + \frac{1}{2.\lambda} \parallel x - u \parallel^2\} \longrightarrow F_\lambda(x) = inf_u\{f(x) + \frac{1}{2.\lambda} \parallel x - u \parallel^2\} \ \forall \lambda \in]0, +\infty[, \forall x \in \Re^N$$

$$\Downarrow$$

$f_n \to f$ in a non classical sense. In fact $f_n \to f$ in the following variational sens:

$\forall x \in \Re^N$, $\forall x_n \to x$, we have $f(x) \leq \underline{lim} f_n(x_n)$ and there exists $z_n \to x$ such that $f_n(z_n) \to f(x)$.

Robustness of convergences and approximations

This convergence is called Epi-convergence ($\stackrel{epi}{\to}$) in the literature and has remarkable stable properties:

- If there exits a bounded sequence $(u_n)_n$ such that $f_n(u_n) \leq inf_{\Re^N} f_n + \epsilon_n$ and $f_n \stackrel{epi}{\to} f$ then $inf_{\Re^N} f_n \to inf_{\Re^N} f$ and if $x_{n_k} \to x'$ then $f(x') = min_{\Re^N} f$
- $f_n \stackrel{epi}{\to} f \leftrightarrow f_n^*(y) \stackrel{epi}{\to} f^*(y) = sup_{x \in \Re^N} \{\langle x, y \rangle - f(x)\}$ (convex and finite dimensional case, convergence of primal problems $\Rightarrow$ convergence of dual problems)
- Epi-convergence is the minimal convergence satisfying the last properties
- Epi-convergence is incomparable with point-wise convergence

Robustness of convergences and approximations

There are many convergences which can be defined in infinite dimensional setting and are more suitable with the study of perturbed problems in parametric optimization, in mechanics, and elasticity as:

- Painlevé-Kuratowski convergence
- Hausdorff convergence
- Mosco convergence
- Slice convergence
- Bounded convergence (or Attouch-Wets convergence)

Robustness of convergences and approximations

We can also define many convergences for sets because the constraint sets in optimization can be also approximated or pertubated:

- $C_n \to C$ if $C = \bigcap_n cl(\bigcup_{k\geq n} C_k) = cl(\bigcup_n \bigcap_{k\geq} C_k)$
- $C_n \to C$ if $d(x, C_n) \to d(x, C)$, $\forall x$
- $C_n \to C$ if $sup_{\|x\|} \leq \rho \mid d(x, C_n) - d(x, C) \mid \to 0$, $\forall \rho$

For regularization of bibariate functions we have:

- $F^n_{\lambda,\mu}(x, y) = inf_{u\in X} sup_{v\in Y}\{L_n(u, v) + \frac{1}{2\lambda} \parallel x - u \parallel^2 - \frac{1}{2\mu} \parallel y - v \parallel^2 \} \to F_{\lambda,\mu}(x, y) = inf_{u\in X} sup_{v\in Y}\{L(u, v) + \frac{1}{2\lambda} \parallel x - u \parallel^2 - \frac{1}{2\mu} \parallel y - v \parallel^2\}$, $\forall (x, y)$, $\forall \lambda$, μ (positive) then $L_n \to L$ in a non classical sense that is:
  - $\forall (x, y)$, $\forall x_n \to x$, $\exists y_n \to y$ such $f(x, y) \leq \underline{\underline{lim}} f_n(x_n, y_n)$
  - $\forall (x, y)$, $\forall y_n \to x$, $\exists x_n \to y$ such $\overline{lim f_n(x_n, y_n)} \leq f(x, y)$
  - $L_n \to L$ in the Epi/hypo-convergence sense
  - if $L_n \to L$ in the Epi/hypo-convergence sense and (x_n, y_n) is a saddle point of L_n (equilibrium point):

    $\forall (x, y)$: $L_n(x_n, y) \leq L_n(x_n, y_n) \leq L_n(x, y_n)$ and $(x_n, y_n) \to (x', y')$ then (x', y') is a saddle point of L and $L_n(x_n, y_n) \to L(x', y')$ when $n \to +\infty$.

Robustness of convergences and approximations

Is- it possible to elaborate a unified approach of general and robust regularization allowing us to stabilize unstable problems and to elaborate efficient hybrid algorithms for approximating a solution of $minf$ **? What is the relationship between the initial problem and its regularized form? What are the fundamental properties of the hard operator** $f \to minf$ **?**

Robustness of convergences and approximations

(P):$min_{x \in C} f(x)$, $C \subset X$ general Hausdorff space.
Now consider a sequence $g, h_k : X \to \Re$ of functions such that $r_k = inf_{x \in C} h_k(x)$ in finite for all $k \geq k_0$ and g is sci. To (P) we associate the following generalized regularization problem $(P_k) : min_{x \in C} F_k(x)$ where $F_k(x) = f(x) + \epsilon_k g(x) + h_k(x)$, $\epsilon_k > 0$ and we suppose that $\epsilon_k \to 0$ if $k \to +\infty$.

Robustness of convergences and approximations

Theorem [Mentagui - 2016, International Journal of Maths Programming]:

Assume that the following conditions hold:

- (a) $i_k = inf_C F_k$ is finite for every $k \geq k_0$ and $(z_k)_k$ be a sequence of C relatively compact satisfying:
$$\frac{F_k(z_k) - i_k}{\epsilon_k} \to 0 \ , \ k \to +\infty$$
- (b) $\frac{h_k(s) - r_k}{\epsilon_k} \to 0 \ , \ k \to +\infty, \ \forall s \in X$
- (c) $S = argmin(f, X) \neq \emptyset$

Then:

- (1) (F_k, X) is stable in the Tikhonov sense
- (2) Any cluster point $\bar{z} \in C$ of $(z_k)_k$ verifies $\bar{z} \in argmin(g, S)$.
- (3) $f(z_k) \to f(\bar{z})$ and $g(z_k) \to g(\bar{z})$ when $k \to +\infty$.
- (4) We have the following asymptotic development:
$$inf_C(f(x) + \epsilon_k g(x) + h_k(x)) =$$
$$min_{x \in C} f(x) + \epsilon_k min_{x \in S} g(x) + inf_{x \in C} h_k(x) + \epsilon_k.\theta_k =$$
$$min_{x \in C} f(x) + \epsilon_k min_{x \in S} g(x) + inf_{x \in S} h_k(x) + \epsilon_k.\theta'_k \ (\theta_k, \theta'_k \to 0).$$
- (5) $\varphi : f \to min(f)$:
$\varphi'(f, g) = lim_{\epsilon \to 0} \frac{min_C(f + \epsilon g) - min_c(f)}{\epsilon \|g\|} = \frac{min_{x \in S} g(x)}{\|g\|}$.

Robustness of convergences and approximations

Remark:

- Our hypotheses are not restrictive and includes all regularizations existing actually in the literature.
- The regularizations $F_\epsilon(x) = f(x) + \epsilon g(x) + h_\epsilon(x)$ have the general form and allow us to construct hybrid algorithms.

Concerning the saddle regularization and saddle point we have:
Consider two general topological Hausdorff spaces X, Y and $f : XxY \to \bar{\Re}$, $g : XxY \to \Re$, $h_\epsilon : XxY \to \Re$ are three functions with $\epsilon > 0$. Each function f, g is assumed to be lower semi-continuous (lsc) at the first variable and upper semi-continuous (usc) at the second variable. Denote by $h_\epsilon^1 = sup_{y \in Y} inf_{x \in X} h_\epsilon(x, y)$ and $h_\epsilon^2 = inf_{x \in X} sup_{y \in Y} h_\epsilon(x, y)$ which are supposed finite for every $\epsilon > 0$ sufficiently small. Assume that the set $S = \{(a, b) \in XxY/(a, b)$ is a saddle point of f$\}$ is nonempty. Set $F_\epsilon(x, y) = f(x, y) + a_\epsilon g(x, y) + h_\epsilon(x, y)$ with $a_\epsilon > 0$, $a_\epsilon \to 0$ when $\epsilon \to 0$. If $h_\epsilon = 0$ and $g(x, y) = a_i \parallel x \parallel^p - b_i \parallel y \parallel^p$ with a_i, b_i are positive real numbers and $p, q \in N^*$ then F_ϵ reduces to the classical Tikhonov regularization.

Robustness of convergences and approximations

Theorem [Mentagui-2016, International Journal of Maths Programming]
Let $(x_\epsilon, y_\epsilon)_\epsilon$ be a relatively compact sequence such that $\alpha_\epsilon = sup_y F_\epsilon(x_\epsilon, y)$, $\beta\epsilon = inf_x F_\epsilon(x, y_\epsilon)$, $\gamma_\epsilon(t) = sup_y h_\epsilon(t, y)$, $\delta_\epsilon(z) = inf_x h_\epsilon(x, z)$ are finite for every ϵ sufficiently small and every $(t, z) \in XxY$. Assume that the following condition holds:

$$lim_{\epsilon\to 0} \frac{\alpha_\epsilon - \beta_\epsilon}{a_\epsilon} = lim_{\epsilon\to 0} \frac{\gamma_\epsilon(t) - \delta_\epsilon(z)}{a_\epsilon} = 0 \ \forall (t, z) \in XxY$$

Then

- (i) any cluster point $(\bar{x}, \bar{y})$ of (x_ϵ, y_ϵ) is a saddle point of f on XxY and is a saddle point of g on S. Furthermore for every $\alpha \in \Re$, there exists a sequence $(\delta_\epsilon^\alpha, \theta_\epsilon^{1,\alpha}, \theta_\epsilon^{2,\alpha}) \to 0_{\Re^3}$ if $\epsilon \to 0$ depending on the scheme under consideration such that $F_\epsilon(x_\epsilon, y_\epsilon) = f(\bar{x}, \bar{y}) + a_\epsilon g(\bar{x}, \bar{y}) + \alpha.h_\epsilon^1 + (1-\alpha)h_\epsilon^2 + a_\epsilon \delta_\epsilon^\alpha$ and the sequence $(g(x_\epsilon, \bar{y}), g(\bar{x}, y_\epsilon), \frac{f(x_\epsilon, \bar{y}) - f(\bar{x}, \bar{y})}{a_\epsilon}, \frac{f(\bar{x}, y_\epsilon) - f(\bar{x}, \bar{y})}{a_\epsilon}, \frac{h_\epsilon^2 - h_\epsilon^1}{a_\epsilon})$ converges to $(g(\bar{x}, \bar{y}), g(\bar{x}, \bar{y}), 0, 0, 0)$ if $\epsilon \to 0$;

Robustness of convergences and approximations

- (ii) $F_\epsilon^i = f(\bar{x}, \bar{y}) + a_\epsilon g(\bar{x}, \bar{y}) + \alpha.h_\epsilon^1 + (1-\alpha)h_\epsilon^2 + a_\epsilon \theta_\epsilon^{i,\alpha}$ and $lim_{\epsilon \to 0} \frac{F_\epsilon^2 - F_\epsilon^1}{a_\epsilon} = 0$ where $F_\epsilon^1 = sup_{y \in Y} inf_{x \in X} F_\epsilon(x, y)$ and $F_\epsilon^2 = inf_{x \in X} sup_{y \in Y} F_\epsilon(x, y)$.
- (iii) (F_ϵ, XxY) is stable in the sense of Tikhonov

Inverse methodology in Perturbation theory (Classical Mechanics, quantum mechanics, optimal control, numerical analysis ...etc)

Perturbation theory comprises mathematical methods for finding an approximate solution to a problem, by starting from the exact solution of a related, simpler problem. A critical feature of the technique is a middle step that breaks the problem into "solvable" and "perturbation" parts. Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

- $\frac{dx(t)}{dt} = f(x,t) + \epsilon.g(x,t,u)$, $u \in U$, $x(0) = x_0$, $\epsilon \simeq 0$ **(Optimal control problems with small parameters as in Missile theory)**
- $\frac{d^2x(t)}{dt^2} + \omega_0^2 x(t) + \epsilon x^3(t) = 0$, $x(0) = x_0$, $\dot{x}(0) = v_0$, $\epsilon \simeq 0$ **(Duffing model in classical mechanics)**

Inverse methodology in Perturbation theory (Classical Mechanics, quantum mechanics, optimal control, numerical analysis ...etc)

Example of resolution:

- (E): $\frac{d^2x(t)}{dt^2} + \frac{x(t)}{\tau} + \frac{\epsilon}{\tau L_0}x^2(t) = 0$, $x(0) = x_0$, $\epsilon \simeq 0$
- solve DE ($\epsilon = 0$): $\frac{d^2x(t)}{dt^2} + \frac{x(t)}{\tau} = 0$, $z(t) = Ae^{-\frac{t}{\tau}}$
- set $x_\epsilon = z(t) + \epsilon x_1(t) + O(\epsilon^2)$ the solution of (E) and put $x_\epsilon(t)$ in (E) then we find:

$$x_\epsilon(t) = x_0(1 - \epsilon\frac{x_0}{L0})e^{-\frac{t}{\tau}} + \epsilon\frac{x_0^2}{L_0}e^{-\frac{2t}{\tau}} + O(\epsilon^2) \Rightarrow x_\epsilon \simeq$$
$$x_0(1 - \epsilon\frac{x_0}{L0})e^{-\frac{t}{\tau}} + \epsilon\frac{x_0^2}{L_0}e^{-\frac{2t}{\tau}} \text{ when } \epsilon \simeq 0$$

- Concerning $\frac{dx(t)}{dt} = f(x,t) + \epsilon.g(x,t,u)$, $u \in U$, $x(0) = x_0$, $\epsilon \simeq 0$ or more generally:

$$\text{(F): } \frac{dx(t)}{dt} = f(x,t,\epsilon,u),\ u \in U,\ x(0) = x_0,\ \epsilon \simeq 0$$

Theorem [Poincaré] : There exists an **analytical solution** $\mathbf{x}_\epsilon(t, u(t))$ **of (F)** with $x_\epsilon(t,u(t)) = x(t,u(t)) + \sum_{i=1}^{\infty} y_i\epsilon^i$, $x(t,u(t))$ is the solution of (F) with $\epsilon = 0$: $\frac{dx(t)}{dt} = f(x,t,0,u)$, $u \in U$, $x(0) = x_0$.

THANK YOU . . .